Instabilities, nonhermiticity and exceptional points in the cranking model

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A cranking harmonic oscillator model, widely used for the physics of fast rotating nuclei and Bose-Einstein condensates, is re-investigated in the context of $\mathcal{PT}$-symmetry. The instability points of the model are identified as exceptional points. It is argued that - even though the Hamiltonian appears hermitian at first glance - it actually is not hermitian within the region of instability.

Quantum instabilities are attracting considerable attention in a variety of physical situations. They can be associated with the formation of solitons and vortices in Bose-Einstein condensates [1], with a sudden change of the moment of inertia of a rotating nucleus (see, for example, Ref.2 and references therein) and a transition from one- to two-dimensional nuclear rotation [3]. A particular example of interest is the Hamiltonian

$$
\begin{aligned}
\hat{H} &= \hbar\omega_1(a_1^\dagger a_1 + \frac{1}{2}) + \hbar\omega_2(a_2^\dagger a_2 + \frac{1}{2}) + \\
&\quad + i\hbar g_1(a_1^\dagger a_2 - a_2^\dagger a_1) - i\hbar g_2(a_1^\dagger a_2^\dagger - a_2 a_1) \quad (1)
\end{aligned}
$$

used in condensed matter physics to describe in a simple way the interaction between an atom and a radiative field [4]. Note that the bi-linear form of (1) corresponds to a linearised version of some more general interactions. As discussed below this may bring about an instability. Higher order terms may or may not remove such instability.

Using the standard relations ($\hbar = m = 1$)

$$
a_k = \sqrt{\frac{\omega_k}{2}}x_k + i\sqrt{\frac{1}{2\omega_k}}p_k \quad (2)
$$

$$
a_k^\dagger = \sqrt{\frac{\omega_k}{2}}x_k - i\sqrt{\frac{1}{2\omega_k}}p_k \quad (3)
$$

where $x_1 = x, x_2 = y$, and choosing special values for the strength constants $g_1 = \Omega(\omega_1 + \omega_2)/2\sqrt{\omega_1\omega_2}$ and $g_2 = \Omega(\omega_2 - \omega_1)/2\sqrt{\omega_1\omega_2}$, one recognises the well-known cranking Hamiltonian (Routhian)

$$
H = \frac{p_x^2}{2} + \frac{\omega_x^2}{2}x^2 + \frac{p_y^2}{2} + \frac{\omega_y^2}{2}y^2 - \Omega L_z \quad (4)
$$

which has been applied in nuclear physics [5, 6] and for rotating Bose-Einstein condensates (cf Ref.7). The Hamiltonian (4) appears as the sum of hermitian operators and is expected - naively at first glance - to be itself a hermitian operator. The same holds when (4) is written in second quantised form (1). In the following we explore the formal character of the instability points of H and $\hat{H}$. We argue that the operators are no longer hermitian at these points, in fact, we show, that these points are exceptional points (EP) [8, 9].

Non-hermitian Hamilton operators have attracted widely spread interest during the recent years (see [10]), be it in the context of effective theories [11], or in the context of finding a hermitian equivalent [12] or in the context of $\mathcal{PT}$-symmetry [13] ($\mathcal{PT}$ is the product of the parity and time reversal operator). One specific aspect of non-hermitian operators are the EPs, being singularities of spectrum and eigenfunctions. As such, they are usually of particular physical significance. They have been discussed in a great variety of physical applications: in optics [14], in mechanics [15], as coalescing resonances [16, 17], in atomic physics [18], and in more theoretical context in $\mathcal{PT}$-symmetric models [19] or in considering their mutual influence [20], to name just a few. In its simplest case they give rise to level repulsion being the more pronounced the nearer they lie to the real axis. Depending on the particular situation they can signal a phase transition [21]. In the present case the EP is associated with the onset of an instability. Note that the Hamiltonian (4) is symmetric under $\mathcal{PT}$-operation irrespective of a special choice of parameters ($\Omega \rightarrow -\Omega$ under $\mathcal{T}$) [22].

It is well known that a Bogoliubov transformation of the Hamiltonian (4)

$$
\begin{pmatrix} q_+ \\ q_- \\ q_-^\dagger \\ q_+^\dagger \end{pmatrix} = \mathcal{B} \begin{pmatrix} a_x \\ a_y \\ a_x^\dagger \\ a_y^\dagger \end{pmatrix} \quad (5)
$$

yields the form (cf [6])

$$
\hat{\mathcal{H}} = \omega_+(q_+^\dagger q_+ + \frac{1}{2}) + \omega_-(q_-^\dagger q_- + \frac{1}{2}) \quad (6)
$$

with the eigenmode energies

$$\omega_\pm^2 = \frac{1}{2}\left(\omega_x^2 + \omega_y^2 + 2\Omega^2 \pm \sqrt{(\omega_x^2 - \omega_y^2)^2 + 8\Omega^2(\omega_x^2 + \omega_y^2)}\,\right). \tag{7}$$

It is also known [6, 22] that ω_-^2 becomes negative when the rotational speed Ω lies between $\min(\omega_x, \omega_y)$ and $\max(\omega_x, \omega_y)$. In the following we assume that $\omega_x > \omega_y$. At the points where the two eigenmodes vanish, that is when $\omega_{-,1} = +\omega_-$ and $\omega_{-,2} = -\omega_-$ coalesce, the matrix $\mathcal{B}$ in (5) becomes singular. This happens at the critical points $\Omega_{c1} = \omega_y$ or $\Omega_{c2} = \omega_x$ signalling an instability.

The coalescence is reminiscent of the behaviour of an EP. To confirm that we are in fact encountering a genuine EP and not a usual degeneracy, we have to analyse the eigenfunctions of the respective Hamiltonians. Of course, this is closely related to the singular behaviour of the Bogoliubov transformation $\mathcal{B}$.

To illuminate both, the underlying physics and the mathematical structure, it is convenient to construct the matrix $\mathcal{U}$ connecting the original canonical coordinates $\vec{p}$ and $\vec{r}$ with the quasi-boson operators q_k and $q_k^\dagger$, that is

$$\begin{pmatrix} p_x \\ p_y \\ x \\ y \end{pmatrix} = \mathcal{U} \begin{pmatrix} q_+ \\ q_- \\ q_-^\dagger \\ q_+^\dagger \end{pmatrix}. \tag{8}$$

As a first step we aim at the generalised classical normal mode coordinates $\vec{P} = (P_+, P_-)$ and $\vec{X} = (X_+, X_-)$ in which H assumes the form

$$\tilde{H} = \frac{P_+^2}{2} + \frac{\omega_+^2}{2}X_+^2 + \frac{P_-^2}{2} + \frac{\omega_-^2}{2}X_-^2. \tag{9}$$

This is achieved by solving the classical equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}p_k = -\frac{\partial H}{\partial r_k} \tag{10}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}r_k = \frac{\partial H}{\partial p_k} \tag{11}$$

which can be written in matrix form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \vec{p} \\ \vec{r} \end{pmatrix} = \mathcal{M}\begin{pmatrix} \vec{p} \\ \vec{r} \end{pmatrix} \tag{12}$$

with

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{pmatrix}\mathcal{H} \tag{13}$$

where

$$\mathcal{H} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & \Omega \\ 0 & 1 & -\Omega & 0 \\ 0 & -\Omega & \omega_x^2 & 0 \\ \Omega & 0 & 0 & \omega_y^2 \end{pmatrix}. \tag{14}$$

Note that (4) can be written as

$$H = \begin{pmatrix} \vec{p} & \vec{r} \end{pmatrix}\mathcal{H}\begin{pmatrix} \vec{p} \\ \vec{r} \end{pmatrix}. \tag{15}$$

The solution of (12) is obtained by exponentiation and reads

$$\begin{pmatrix} \vec{p}(t) \\ \vec{r}(t) \end{pmatrix} = \mathcal{U}\exp(\mathcal{D}\,t)\mathcal{V}\begin{pmatrix} \vec{p}(0) \\ \vec{r}(0) \end{pmatrix} \tag{16}$$

where $\mathcal{D} = \mathrm{diag}(-i\omega_+, -i\omega_-, i\omega_-, i\omega_+)$ is the diagonal form of $\mathcal{M} = \mathcal{U}\mathcal{D}\mathcal{V}$ containing the eigenmodes. The columns of $\mathcal{U}$ and $\mathcal{V}$ are the right hand and left hand eigenvectors, respectively, of $\mathcal{M}$. Note that the eigenmodes are obtained from the non-symmetric matrix $\mathcal{M}$; from this classical view point it is therefore no surprise that some of the eigenvalues occurring in (6) may be complex. As the column vectors of $\mathcal{U}$ and $\mathcal{V}$ form a biorthogonal system, we can choose $\mathcal{V} = \mathcal{U}^{-1}$. Also, we observe from the special form of (13) that

$$\mathcal{V} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & +i & 0 & 0 \\ +i & 0 & 0 & 0 \end{pmatrix}\mathcal{U}^T\begin{pmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{17}$$

While the explicit form of $\vec{P}(t)$ and $\vec{X}(t)$ is of little interest the essential point here is the classical instability occurring for negative values of ω_-^2, that is for $\omega_y \leq \Omega \leq \omega_x$. In fact, the harmonic oscillator potential has the 'wrong' sign in (9) for the coordinates $P_-(t)$ and $X_-(t)$. From (16) we read off the classical 'run away' solution in this parameter range yielding the $\sim \exp(|\omega_-|t)$ behaviour for position and momentum. The corresponding quantum mechanical behaviour is discussed below.

Using the form (15) we aim at a form corresponding to (6), *viz.*

$$\hat{H} = \begin{pmatrix} q_+ & q_- & q_-^\dagger & q_+^\dagger \end{pmatrix}H_{QM}\begin{pmatrix} q_+ \\ q_- \\ q_-^\dagger \\ q_+^\dagger \end{pmatrix} \tag{18}$$

with

$$H_{QM} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & \omega_+ \\ 0 & 0 & \omega_- & 0 \\ 0 & \omega_- & 0 & 0 \\ \omega_+ & 0 & 0 & 0 \end{pmatrix}. \tag{19}$$

From (8) this implies that $\mathcal{U}$ must be normalised such that $\mathcal{H} = \mathcal{U}^T H_{QM}\mathcal{U}$. The explicit form of the matrix elements of $\mathcal{U}$ are given in [23], however the the quoted paper focuses upon significantly smaller values of Ω than the range of instability. The analytic form allows pertinent statements in general, and in particular an expansion in Ω around the critical points $\Omega_{c1} = \omega_y$ and $\Omega_{c2} = \omega_x$.
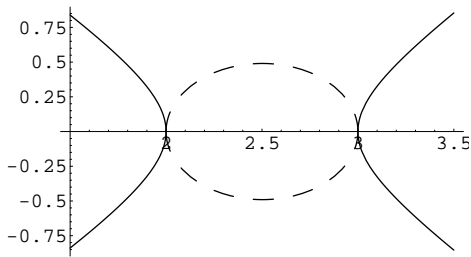
FIG. 1: Relevant spectrum $\pm\omega_-$ as a function of Ω in arbitrary units. Parameters chosen are $\omega_x = 3$ and $\omega_y = 2$. The dashed lines indicate the imaginary part.

The essential results are as follows:

(i) When $\Omega \notin [\omega_y, \omega_x]$ the commutators $[q_j, q_k^\dagger] = \delta_{j,k}$ follow from $[r_m, p_n] = i\delta_{m,n}$. It guarantees that the boson operators are creation and annihilation operators for the excitations, in the present case, with energies ω_- and ω_+. However, this holds *only* when ω_-^2 is positive; if ω_-^2 is negative ($\Omega \in [\omega_y, \omega_x]$) the commutator $[q_-, q_-^\dagger]$ is negative with the implication that the operators q_- and $q_-^\dagger$ are no longer proper boson operators. We further note that $\mathcal{V}$ - as given by (17) - no longer is the inverse of $\mathcal{U}$ for this parameter range.

(ii) The end points of the instability region, i.e. the points $\Omega_{c1} = \omega_y$ and $\Omega_{c2} = \omega_x$ can be clearly identified as EPs. In fact, while the two eigenvectors associated with the two distinct eigenvalues $+\omega_-$ and $-\omega_-$ are obviously linearly independent, they become aligned, i.e. linearly dependent, at Ω_{c1} and Ω_{c2} where ω_- vanishes; this is the *clear signature* of an EP [24]. We recall: a genuine degeneracy would have two linearly independent eigenvectors. EPs are a universal phenomenon occurring in spectra and eigenfunctions under variation of parameters. For hermitian operators they can occur only when such parameters are continued into the complex plane thus rendering the original hermitian operator effectively nonhermitian.

EPs are square root singularities of the spectrum: in the present case the spectrum has a branch cut in Ω ranging from ω_y to ω_x. When the eigenvalues $+\omega_-$ and $-\omega_-$ are continued beyond the EP, they become imaginary (as was also noticed in [22]) for $\Omega \in [\omega_y, \omega_x]$, again with opposite sign (see Fig.1); clearly this contradicts H being hermitian for this parameter range.

(iii) The correct normalisation enforced by (8) (to guarantee the correct commutation relations when $\Omega \notin [\omega_y, \omega_x]$) has the consequence that the leading terms of the components of the critical eigenvectors behave as $(\Omega - \Omega_c)^{-1/4}$ when approaching the critical point. This particular singular behaviour - the forth root and the infinity - is again a consequence of the eigenfunctions at an EP [25]. In fact, it has been shown in general [26] that the scalar product of the two eigenfunctions - associated with the two coalescing levels - must vanish as a square root, in the present case as $(\Omega - \Omega_c)^{1/2}$. As a consequence, when normalisation is enforced by dividing by the square root of the scalar product, the singular behaviour follows as indicated. Moreover, the forth root has the consequence that - for the wave function - a clockwise encircling of the EP in the Ω-plane yields a result that has a phase that is different from that of a counterclockwise encirclement. In fact, considering $\sqrt[4]{z}$ (taking $z = \Omega - \Omega_c$), one obtains $+i$ when z has described a full counterclockwise circle around zero and $-i$ when going in the opposite direction. This particular Riemann sheet structure has been experimentally established in microwave cavities [27]. It would be a challenge to confirm it in the present context with a BEC or with Raman scattering using an incident laser beam upon vibrational modes of a medium.

So far, we have established the seemingly surprising result that the Hamilton operator (4) - or its second quantised counterpart - fails to be hermitian when $\omega_y \leq \Omega \leq \omega_x$. The endpoints of this interval are EPs. We stress that this result is based on an analytic continuation obtained from the range $\Omega < \omega_y$, or equally, from the range $\Omega > \omega_x$. These two (hermitian) ranges are of course also analytically connected.

It appears apposite to contrast our findings with common wisdom about the solutions of the Schroedinger equation of (9). In fact, if (9) is considered in isolation, the Hamiltonian appears perfectly hermitian, also for $\omega_-^2 < 0$. It has a continuous spectrum associated with the unbounded classical motion in the coordinates P_- and X_-. The quantum mechanical wave function is asymptotically of the form $\exp(i|\omega_-|X_-^2/2)$ apart from a hypergeometric function. The crucial aspect explaining this apparent discrepancy lies in the transformation that brings us from (4) to (9). As long as $\Omega \notin [\omega_y, \omega_x]$ the two operators are equivalent up to a similarity transformation. For $\Omega \in [\omega_y, \omega_x]$ they are not. And the transformation breaks down exactly at the EPs, the singularity that signals the instability point.

This is a beautiful demonstration of a $\mathcal{PT}$-symmetric operator [13], yet with a special twist: (4) appears hermitian to the naked eye, but its spectrum is not real when $\Omega \in [\omega_y, \omega_x]$. While the operator is $\mathcal{PT}$-symmetric, the symmetry is broken by the state vector. Thus, in this parameter range the hermitian form (9) is not its hermitian equivalent.

In conclusion, we mention that the two exceptional points collapse into a diabolic point [28] when $\omega_y = \omega_x$; in this case $\Omega = \omega_x$ is a regular point with a genuine degeneracy for $\omega_- = 0$.

Acknowledgements

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